

FREE SYMMETRIC ALGEBRAS IN DIVISION RINGS GENERATED BY ENVELOPING ALGEBRAS OF LIE ALGEBRAS

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ABSTRACT. For any Lie algebra L over a field, its universal enveloping algebra $U(L)$ can be embedded in a division ring $\mathfrak{D}(L)$ constructed by Lichtman. If $U(L)$ is an Ore domain, $\mathfrak{D}(L)$ coincides with its ring of fractions.

It is well known that the principal involution of L , $x \mapsto -x$, can be extended to an involution of $U(L)$, and Cimprič has proved that this involution can be extended to one on $\mathfrak{D}(L)$.

For a large class of noncommutative Lie algebras L over a field of characteristic zero, we show that $\mathfrak{D}(L)$ contains noncommutative free algebras generated by symmetric elements with respect to (the extension of) the principal involution. This class contains all noncommutative Lie algebras such that $U(L)$ is an Ore domain.

INTRODUCTION

A long-standing conjecture of Makar-Limanov states that a division ring which is finitely generated (as a division ring) and infinite dimensional over its center contains a free (associative) algebra of rank two over the center [19]. Makar-Limanov's conjecture has been extensively investigated, see e.g. [9]. We remark that containing a free algebra over the center is equivalent to containing a free algebra over any central subfield [21, Lemma 1].

In [7], we raised a question related to Makar-Limanov's conjecture. In the presence of an involution, $x \mapsto x^*$, a natural investigation would be whether a division ring satisfying Makar-Limanov's conjecture contains a free algebra of rank 2 generated by symmetric elements i.e. elements satisfying $x^* = x$.

Let k be a field. There are two important families of k -algebras which are endowed with natural involutions. The first one is the class of group algebras $k[G]$ over a group G . The map $k[G] \rightarrow k[G]$, $\sum_{x \in G} x a_x \mapsto \sum_{x \in G} x^{-1} a_x$, where, for all $x \in G$, a_x are elements of k which are zero except for a finite number of x , is called the *canonical involution*. The second important class of k -algebras endowed with an involution is the class of the universal enveloping algebras of Lie algebras. Let L be a Lie k -algebra and $U(L)$ be its universal

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enveloping algebra. There is a natural involution on L , $L \rightarrow L$, $x \mapsto -x$. It is well known that this involution can be extended to an involution of $U(L)$ that fixes k elementwise. We call this involution the *principal involution* on L .

It is not known for which groups G the group algebra $k[G]$ is embeddable in a division ring. Moreover, even if $k[G]$ is embedded in a division ring D , the canonical involution may not be extended to one on D . When G is an orderable group, the group ring $k[G]$ is embeddable in $k((G))$, the division algebra of Malcev-Neumann series of G over k [22], [23]. Let $k(G)$ be the division algebra of $k((G))$ generated by $k[G]$. In [24], it was shown that if G is not abelian, any division ring containing $k[G]$ must contain a free k -algebra of rank two. In [7], we proved that the canonical involution of $k[G]$ can be extended to $k(G)$ and that $k(G)$ contains a free k -algebra of rank two generated by symmetric elements with respect to the canonical involution.

That $U(L)$, the universal enveloping algebra of a Lie k -algebra L , can be embedded in a division algebra was proved by Cohn in [5]. In [15], Lichtman obtained a simpler proof of this fact. For each Lie k -algebra L , he constructed a division k -algebra $\mathfrak{D}(L)$ that contains $U(L)$ and it is generated by it. In the study of division rings, $\mathfrak{D}(L)$ plays a similar role to $k(G)$ for the group algebra $k[G]$ of an ordered group G . In [3], Cimprić proved that the principal involution on $U(L)$ can be extended to $\mathfrak{D}(L)$. In [16], Lichtman also proved that if k is a field of characteristic zero and L is not commutative, any division ring that contains $U(L)$ must contain a free algebra of rank 2. The main aim of this paper is to show that if k is of characteristic zero, then, for a large class of noncommutative Lie k -algebras L , the division ring $\mathfrak{D}(L)$ contains a free k -algebra of rank two generated by symmetric elements with respect to the principal involution. This class contains the noncommutative residually nilpotent Lie k -algebras and the noncommutative Lie k algebras L for which $U(L)$ is an Ore domain. We give explicit generators for the free algebras generated by symmetric elements.

The contents of Section 1 are known. We sketch the construction of $\mathfrak{D}(L)$ following Lichtman [15], and show how the principal involution is extended to $\mathfrak{D}(L)$ as shown in [3]. We also present some results on the completion of skew polynomial rings that will be needed in Section 3.

Let k be a field of characteristic zero. Let H be the Heisenberg Lie k -algebra, i.e. the Lie k -algebra with presentation

$$H = \langle x, y \mid [[y, x], x] = [[y, x], y] = 0 \rangle.$$

The aim of Section 2 is to find generators of a free k -algebra of rank two generated by symmetric elements with respect to the principal involution inside $\mathfrak{D}(H)$, the Ore division ring of fractions of $U(H)$. These generators must be of a certain form so that the results of Sections 3 and 4 can be applied. The generators are obtained using a result of Cauchon [2], who used the techniques from [20], and applying a technical result proved in Subsection 2.1 that symmetrizes the free algebra obtained from Cauchon's result.

In Section 3, we obtain a free algebra of rank two generated by symmetric elements with respect to the principal involution inside $\mathfrak{D}(L)$ where L is a (certain generalization of) residually nilpotent Lie algebra over a field k of characteristic zero. This free algebra

is obtained by pulling back the free algebra generated by symmetric elements in the case of the Heisenberg Lie algebra. This pullback is done using a series technique which can be seen as a translation of the one used for the group algebra $k[G]$ of an ordered group G and $k(G)$, in [24] and [7], to the language of universal enveloping algebras and $\mathfrak{D}(L)$.

In Section 4, we use the technique from [16] to obtain a free algebra inside $\mathfrak{D}(L)$, which is generated by symmetric elements with respect to the principal involution, from the one obtained in the residually nilpotent case.

In Section 5 further comments on the subject are made.

All algebras (except for Lie algebras) will be associative with 1, and their homomorphisms will be understood to respect 1. We will also work with Lie algebras, but the term Lie will always be explicit so that no confusion is possible.

A *valuation* on an algebra A is a map $\vartheta: A \rightarrow \mathbb{Z} \cup \{\infty\}$ that satisfies the following three properties (i) $\vartheta(x) = \infty$ if and only if $x = 0$, (ii) $\vartheta(xy) = \vartheta(x) + \vartheta(y)$, and (iii) $\vartheta(x + y) \geq \min\{\vartheta(x), \vartheta(y)\}$, for all $x, y \in A$.

By *free algebra on a set X* over a field k , we mean the associative k -algebra of noncommuting polynomials with indeterminates from the set X .

Given a k -algebra A , a *k -derivation* is a k -linear map $\delta: A \rightarrow A$ such that $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in A$.

We shall give a few words about Ore localization. Notice that we shall only need localization of domains at certain multiplicative sets. For further details the reader is referred to, for example, [13].

We say that an algebra R is a *domain* if it is nonzero and contains no zero divisors other than 0.

A subset \mathfrak{S} of R is *multiplicative* if it contains 1 and is closed under multiplication, i.e. $x, y \in \mathfrak{S}$ implies that $xy \in \mathfrak{S}$. A multiplicative subset of the domain R is a *left Ore* subset if for any $a \in R$ and $s \in \mathfrak{S}$, $\mathfrak{S}a \cap Rs \neq \emptyset$. *Right Ore* subsets are defined similarly. A left and right Ore subset of R is called an *Ore subset*. We say that a domain R is an *Ore domain* if the multiplicative set $R \setminus \{0\}$ is an Ore subset of R .

Let R, R' be any algebras and \mathfrak{S} a subset of R . An algebra homomorphism $f: R \rightarrow R'$ is said to be *\mathfrak{S} -inverting* if f maps \mathfrak{S} into the units of R' . We say that a \mathfrak{S} -inverting homomorphism $f: R \rightarrow R'$ is a *universal \mathfrak{S} -inverting homomorphism* if for any other \mathfrak{S} -inverting homomorphism $g: R \rightarrow R''$, there exists a unique homomorphism $\tilde{g}: R' \rightarrow R''$ such that $g = \tilde{g}f$.

The results we will need are the following. Let R be a domain and \mathfrak{S} a (left) Ore subset in R . There exists a unique algebra $\mathfrak{S}^{-1}R$ such that R embeds into $\mathfrak{S}^{-1}R$, all the elements of $\mathfrak{S}^{-1}R$ can be expressed in the form $s^{-1}r$ where $a \in R$, $s \in \mathfrak{S}$, and the embedding $R \hookrightarrow \mathfrak{S}^{-1}R$ is universal \mathfrak{S} -inverting. Moreover, if $\mathfrak{S} = R \setminus \{0\}$, then $\mathfrak{S}^{-1}R$ is a division algebra that contains R and is generated by it. In this event, we will say that $\mathfrak{S}^{-1}R$ is the *Ore division ring of fractions* of R .

1. PRELIMINARIES

In this section, we present known results and definitions that will be useful in later sections. The lesser known results are proved in some detail for the sake of completeness.

1.1. The universal enveloping algebra $U(L)$ and the division ring $\mathfrak{D}(L)$. Throughout this section, k will denote a field.

Let L be a Lie k -algebra. We will denote by $U(L)$ its *universal enveloping algebra*, see [11] or [6] for further details.

Recall that one can obtain a Lie k -algebra from any associative k -algebra A . Indeed, define a new multiplication on A by the rule $[x, y] = xy - yx$. The k -vector space A with this multiplication is a Lie k -algebra denoted by A^- . The universal enveloping algebra of a Lie k -algebra L is an associative k -algebra $U(L)$, such that L is a Lie subalgebra of $U(L)^-$, satisfying the property that given an associative k -algebra A , any Lie algebra homomorphism $L \rightarrow A^-$ can be extended to a homomorphism $U(L) \rightarrow A$.

In particular one obtains that any homomorphism of Lie k -algebras $L \rightarrow L'$ can be uniquely extended to a homomorphism of associative k -algebras $U(L) \rightarrow U(L')$. From this universal property one also obtains the augmentation map $\varepsilon: U(L) \rightarrow k$, which is the unique extension of the Lie k algebra homomorphism $L \rightarrow k^-$, $x \mapsto 0$.

Let L be a Lie k -algebra. Let \mathcal{L} be a k -linearly independent subset of L . Suppose that we have defined a total order $<$ in \mathcal{L} . The set of *standard monomials* in \mathcal{L} is the subset of $U(L)$ consisting of the monomials of the form $x_1 x_2 \cdots x_m$ with $m \geq 0$, $x_i \in \mathcal{L}$ and $x_1 \leq x_2 \leq \cdots \leq x_m$, where we understand that the identity element in $U(L)$ is the standard monomial corresponding to $m = 0$.

Let $\mathcal{B} = \{x_i \mid i \in I\}$ be a totally ordered basis of L . The Poincaré-Birkhoff-Witt Theorem (PBW Theorem for short) states that the standard monomials in \mathcal{B} form a k -basis of $U(L)$. Thus, every element of $U(L)$ can be uniquely expressed as a finite sum $\sum x_{i_1} x_{i_2} \cdots x_{i_r} a_{i_1 i_2 \cdots i_r}$, where $x_{i_1} x_{i_2} \cdots x_{i_r}$ is a standard monomial in \mathcal{B} and $a_{i_1 i_2 \cdots i_r} \in k$.

Let L be a Lie k -algebra and $U(L)$ its universal enveloping algebra. In [15], Lichtman gives a construction of a skew field $\mathfrak{D}(L)$ which contains $U(L)$ and it is generated by it. We recall the construction (without proofs) for later reference. For further details, the interested reader is referred to [4, Section 2.6] or the original paper [15]. Most of our exposition is taken from [3, Section 2].

The *standard filtration*

$$\cdots \subseteq F_1 \subseteq F_0 \subseteq F_{-1} \subseteq F_{-2} \subseteq \cdots$$

of $U(L)$ is defined by $F_i = \{0\}$ if $i > 0$, $F_0 = k$ and for $i = -n < 0$, F_i is the vector subspace of $U(L)$ generated by all products containing $\leq n$ elements from L . The mapping $\vartheta: U(L) \rightarrow \mathbb{Z} \cup \{\infty\}$ defined by $\vartheta(f) = \sup\{i \mid f \in F_i\}$ is a valuation called the *standard valuation*. Let $U(L)[t, t^{-1}]$ be the ring of Laurent polynomials in a central variable t . Extend ϑ to a valuation on $U(L)[t, t^{-1}]$ by $\vartheta(\sum_i t^i f_i) = \min\{\vartheta(f_i) + i \mid i \in \mathbb{Z}\}$. Define $T = \{h \in U(L)[t, t^{-1}] \mid \vartheta(h) \geq 0\} \subseteq U(L)[t]$. Note that $U(L)[t, t^{-1}]$ is canonically isomorphic to $\mathcal{C}^{-1}T$, the Ore localization of T at the multiplicative set $\mathcal{C} = \{1, t, t^2, \dots\}$. For every $n = 1, 2, \dots$,

write $T_n = T/t^nT$ and \mathcal{U}_n for the projection of $\mathcal{U} = T \setminus tT$ onto T_n . It turns out that \mathcal{U}_n is a regular Ore set in T_n for every n and we denote by $S_n = \mathcal{U}_n^{-1}T_n$ its Ore localization. There are natural surjective k -algebra homomorphisms

$$\cdots \rightarrow S_{n+1} \rightarrow S_n \rightarrow \cdots \rightarrow S_1$$

Write S for the inverse limit of S_n . The localization $D = \mathcal{C}^{-1}S$ is a division ring. There exists a natural embedding of T into S which extends to a natural embedding of $U(L)[t, t^{-1}] = \mathcal{C}^{-1}T$ into $D = \mathcal{C}^{-1}S$. Write $\mathfrak{D}(L)$ for the minimal division ring of D containing $U(L)$. We remark that if $U(L)$ is an Ore domain, then $\mathfrak{D}(L)$ is the Ore division ring of fractions of $U(L)$. Indeed, the universal property of Ore localizations implies that there exists a k -algebra homomorphism from the Ore division ring of fractions of $U(L)$, but $\mathfrak{D}(L)$ is a division ring that contains $U(L)$ and is generated by it. Thus they are isomorphic.

Now we turn our attention to involutions.

Let R be an associative k -algebra. A k -linear map $*$: $R \rightarrow R$ is a k -involution if for all $x, y \in R$, $(xy)^* = y^*x^*$, $x^{**} = x$.

The analogous concept for Lie algebras is as follows. Let L be a Lie k -algebra. A k -linear map $*$: $L \rightarrow L$ is a k -involution if for all $x, y \in L$, $[x, y]^* = [y^*, x^*]$, $x^{**} = x$. The main example of a k -involution in a Lie k -algebra is what we call the *principal involution*. It is defined by $x \mapsto -x$ for all $x \in L$.

The following proposition is a slight variation of [3, Proposition 5] whose proof we follow.

Proposition 1.1. *Every k -involution on a Lie k -algebra L has a canonical extension to $\mathfrak{D}(L)$. In particular, the principal involution of L can be canonically extended to a k -involution of $\mathfrak{D}(L)$.*

Proof. Every k -involution extends uniquely from L to its enveloping algebra $U(L)$ [6, Section 2.2.17]. Setting $t^* = t$, we get an involution on $U(L)[t, t^{-1}]$ which induces an involution on T . Since $(t^nT)^* \subseteq t^nT$ for every n , we have an induced involution on T/t^nT which will also be denoted by $*$. Note that the natural epimorphisms $\phi_n : T_{n+1} \rightarrow T_n$ are $*$ -homomorphisms.

One can prove that $\mathcal{U}^* \subseteq \mathcal{U}$ and $\mathcal{U}_n^* \subseteq \mathcal{U}_n$. Then the involution on T_n extends uniquely to an involution of $S_n = \mathcal{U}_n^{-1}T_n$. It is easy to verify that the natural epimorphisms $\phi'_n : S_{n+1} \rightarrow S_n$ are $*$ -homomorphisms. It follows that the termwise involution on the inverse system of S_n and ϕ'_n induces an involution on its inverse limit S . Since $t^* = t$, this involution extends uniquely to an involution on $D = \mathcal{C}^{-1}S$. Since $\mathfrak{D}(L) \cap \mathfrak{D}(L)^*$ is a division subring of D containing $U(L) = U(L)^*$ and since $\mathfrak{D}(L)$ is the smallest subfield of D containing $U(L)$, it follows that $\mathfrak{D}(L) = \mathfrak{D}(L)^*$. \square

Some other important properties of $\mathfrak{D}(L)$ that we will need in Section 3 are contained in the following result [17, Proposition 2.5].

Proposition 1.2. *Let L be a Lie k -algebra. Suppose that N is a Lie subalgebra of L . The following properties are satisfied:*

(1) *The following diagram is commutative*

$$\begin{array}{ccc} U(N) & \hookrightarrow & \mathfrak{D}(N) \\ \downarrow & & \downarrow \\ U(L) & \hookrightarrow & \mathfrak{D}(L) \end{array}$$

(2) *If \mathcal{B}_N is a basis of N and \mathcal{C} is a set of elements of $L \setminus N$ such that $\mathcal{B}_N \cup \mathcal{C}$ is a basis of L , then the standard monomials in \mathcal{C} are linearly independent over $\mathfrak{D}(N)$.* \square

1.2. Formal differential and pseudo-differential operator algebras. Let R be a k -algebra, δ a k -derivation on R and σ a k -algebra automorphism of R . The *formal differential operator algebra* over R (respectively, the *skew polynomial algebra* over R), denoted by $R[x; \delta]$ (respectively, $R[x; \sigma]$), is defined as the k -algebra S such that

- (a) S is a ring containing R as a k -subalgebra.
- (b) x is an element of S .
- (c) S is a free right R -module with basis $\{1, x, x^2, \dots\}$.
- (d) $ax = xa + \delta(a)$ for all $a \in R$ (respectively, $ax = x\sigma(a)$ for all $a \in R$.)

The k -algebra $R[x; \delta]$ has the following universal property. Given a k -algebra T , a k -algebra homomorphism $\phi: R \rightarrow T$ and an element $y \in T$ such that

$$\phi(a)y = y\phi(a) + \phi(\delta(a)) \quad \text{for all } a \in R, \quad (1.1)$$

there exists a unique k -algebra homomorphism $\psi: R[x; \delta] \rightarrow T$ such that $\psi|_R = \phi$ and $\psi(x) = y$. There is an analogous universal property for $R[x; \sigma]$, but we will not need it.

Given $R[x; \delta]$, one can construct the *formal pseudo-differential operator ring*, denoted $R((t_x; \delta))$, consisting of the formal Laurent series $\sum_{i=n}^{\infty} t_x^i a_i$, with $n \in \mathbb{Z}$ and coefficients $a_i \in R$, satisfying $at_x^{-1} = t_x^{-1}a + \delta(a)$ for all $a \in R$. Therefore

$$at_x = t_x a - t_x \delta(a) t_x = \sum_{i=1}^{\infty} t_x^i (-1)^{i-1} \delta^{i-1}(a), \quad (1.2)$$

for any $a \in R$.

The subset $R[[t_x; \delta]]$ of $R((t_x; \delta))$ consisting of the Laurent series of the form $\sum_{i=0}^{\infty} t_x^i a_i$ is a k -subalgebra of $R((t_x; \delta))$.

The ring $R[[t_x; \delta]]$ can be regarded in another way. Define $R\langle t_x; \delta \rangle$ as the k -algebra $R\langle t_x \mid at_x = t_x a - t_x \delta(a) t_x, a \in R \rangle$. In other words, the k -algebra $R\langle t_x; \delta \rangle$ is isomorphic to the coproduct $R \coprod_k k[t_x]$ modulo the two-sided ideal generated by $\{at_x = t_x a - t_x \delta(a) t_x, a \in R\}$. By definition, the k -algebra $R\langle t_x; \delta \rangle$ has the following universal property. Suppose that we have a k -algebra T , a k -algebra homomorphism $\phi: R \rightarrow T$ and an element $t_y \in T$ such that

$$\phi(a)t_y = t_y\phi(a) - t_y\phi(\delta(a))t_y \quad \text{for all } a \in R. \quad (1.3)$$

Then there is a unique k -algebra homomorphism $\psi: R\langle t_x; \delta \rangle \rightarrow T$ such that $\psi|_R = \phi$ and $\psi(t_x) = t_y$.

Let $\varepsilon_1: k[t_x] \rightarrow R\langle t_x; \delta \rangle$ and $\varepsilon_2: R \rightarrow R\langle t_x; \delta \rangle$ be the natural homomorphisms of k -algebras. By the universal property, there is a k -algebra homomorphism $\varphi: R\langle t_x; \delta \rangle \rightarrow R[[t_x; \delta]]$ such that, for any $n \geq 1$, $\varphi(\varepsilon_1(t_x^n)) = t_x^n$ and, for any $a \in R$, $\varphi(\varepsilon_2(a)) = a$.

Therefore ε_1 and ε_2 are injective homomorphisms. To simplify the notation, we just identify R and $k[t_x]$ with their images in $R\langle t_x; \delta \rangle$ without making any reference to the embeddings ε_1 and ε_2 .

Given a positive integer n , any element $f \in R\langle t_x; \delta \rangle$ can be expressed as

$$f = a_0 + t_x a_1 + \cdots + t_x^{n-1} a_{n-1} + t_x^n f_n,$$

where $f_n \in R\langle t_x; \delta \rangle$, and unique $a_0, \dots, a_{n-1} \in R$. Hence, the k -algebra $R[[t_x \delta]]$ is the completion of $R\langle t_x; \delta \rangle$ with respect to the topology induced by the chain of ideals $\{t_x^n R\langle t_x; \delta \rangle\}_{n \geq 0}$ (see [10, Lemma 7.3] for more details). Moreover the set $\mathcal{S} = \{1, t_x, t_x^2, \dots\}$ is a left denominator set of $R[[t_x; \delta]]$ such that the Ore localization $\mathcal{S}^{-1}R[[t_x; \delta]]$ is the k -algebra $R((t_x; \delta))$, see for example [4, Theorem 2.3.1].

Note that there is a natural embedding $R[x; \delta] \hookrightarrow R((t_x; \delta_x))$ sending x to t_x^{-1} .

The following lemma will be useful.

Lemma 1.3. *Let R and S be two k -algebras. Suppose that $\delta_w: R \rightarrow R$ and $\delta_z: S \rightarrow S$ are k -derivations of R and S , respectively. Let $\phi: R \rightarrow S$ be a k -algebra homomorphism such that*

$$\phi(\delta_w(a)) = \delta_z(\phi(a)) \quad \text{for all } a \in R. \quad (1.4)$$

Then

- (1) ϕ can be extended to a unique k -algebra homomorphism

$$R[w; \delta_w] \rightarrow S[z; \delta_z]$$

sending w to z .

- (2) ϕ can be extended to a unique k -algebra homomorphism

$$\psi: R\langle t_w; \delta_w \rangle \rightarrow S\langle t_z; \delta_z \rangle,$$

such that $\psi(t_w) = t_z$.

- (3) *The k -algebra homomorphism $\psi: R\langle t_w; \delta_w \rangle \rightarrow S\langle t_z; \delta_z \rangle$ induces a k -algebra homomorphism*

$$\Psi: R((t_w; \delta_w)) \rightarrow S((t_z; \delta_z)), \quad \sum_i t_w^i a_i \mapsto \sum_i t_z^i \phi(a_i).$$

Proof. (1) follows from the universal property of $R[w; \delta_w]$.

(2) follows from the universal property of $R\langle t_w; \delta_w \rangle$.

(3) The homomorphism ψ is such that $\psi(a) = \phi(a)$ for all $a \in R$ and $\psi(t_w) = t_z$. Hence it induces the commutative diagram of k -algebra homomorphisms

$$\begin{array}{ccc} \frac{R\langle t_w; \delta_w \rangle}{t_w^{n+1} R\langle t_w; \delta_w \rangle} & \longrightarrow & \frac{S\langle t_z; \delta_z \rangle}{t_z^{n+1} S\langle t_z; \delta_z \rangle} \\ \downarrow & & \downarrow \\ \frac{R\langle t_w; \delta_w \rangle}{t_w^n R\langle t_w; \delta_w \rangle} & \longrightarrow & \frac{S\langle t_z; \delta_z \rangle}{t_z^n S\langle t_z; \delta_z \rangle} \end{array}$$

where the vertical arrows are the canonical projections and the horizontal arrows are induced by ψ in the natural way.

The universal property of the completion yields the k -algebra homomorphism $\Phi: R[[t_w; \delta_w]] \rightarrow S[[t_z; \delta_z]]$ with $\Phi(\sum_{i=0}^{\infty} t_w^i a_i) = \sum_{i=0}^{\infty} t_z^i \phi(a_i)$. Since $R((t_w; \delta_w))$ is the Ore localization of $R[[t_w; \delta_w]]$ at $\mathcal{S} = \{1, t_w, t_w^2, \dots\}$, we get the desired extension. \square

A slight variation of the next result can be found in [10].

Lemma 1.4. *Let R be a k -algebra, let $\delta_w: R \rightarrow R$ be a k -derivation on R and consider the formal differential operator algebra $R[w; \delta_w]$. Let S be a k -algebra. Consider the inner k -derivation, defined by an element $s \in S$, $\delta_s: S \rightarrow S$ given by $\delta_s(b) = bs - sb$ for all $b \in S$.*

Suppose that the following properties hold:

- (a) $R[w; \delta_w]$ is a subring of S ,
- (b) $\delta_s(R) \subseteq R$, and
- (c) $\delta_s(w) \in R$.

Then the following assertions hold true.

- (1) δ_s induces a k -derivation $\delta_s: R\langle t_w; \delta_w \rangle \rightarrow R\langle t_w; \delta_w \rangle$ such that $\delta_s(t_w) = -t_w \delta_s(w) t_w$.
- (2) δ_s induces a k -derivation $\delta_s: R((t_w; \delta_w)) \rightarrow R((t_w; \delta_w))$ such that $\delta_s\left(\sum_{i \geq 0} t_w^i a_i\right) = \sum_{i \geq 0} \delta_s(t_w^i a_i)$, where $\delta_s(t_w) = -t_w \delta_s(w) t_w = \sum_{i \geq 1} t_w^{i+1} (-1)^i \delta_w^{i-1}(\delta_s(w))$, and $\delta_s(t_w^{-1}) = \delta_s(w)$.

Proof. (1) We must show that there exists a homomorphism of k -algebras $\Phi: R\langle t_w; \delta_w \rangle \rightarrow \mathbb{T}_2(R\langle t_w; \delta_w \rangle)$, $f \mapsto \begin{pmatrix} f & \delta_s(f) \\ 0 & f \end{pmatrix}$, where $\mathbb{T}_2(R\langle t_w; \delta_w \rangle)$ is the ring of 2×2 upper triangular matrices over $R\langle t_w; \delta_w \rangle$.

By the universal property (1.3), this is equivalent to proving that, for any $a \in R$, the following matrix equality holds

$$\begin{pmatrix} a & \delta_s(a) \\ 0 & a \end{pmatrix} \begin{pmatrix} t_w & \delta_s(t_w) \\ 0 & t_w \end{pmatrix} = \begin{pmatrix} t_w & \delta_s(t_w) \\ 0 & t_w \end{pmatrix} \begin{pmatrix} a & \delta_s(a) \\ 0 & a \end{pmatrix} - \begin{pmatrix} t_w & \delta_s(t_w) \\ 0 & t_w \end{pmatrix} \begin{pmatrix} \delta_w(a) & \delta_s(\delta_w(a)) \\ 0 & \delta_w(a) \end{pmatrix} \begin{pmatrix} t_w & \delta_s(t_w) \\ 0 & t_w \end{pmatrix},$$

which yields

$$\begin{pmatrix} at_w & \delta_s(a)t_w + a\delta_s(t_w) \\ 0 & at_w \end{pmatrix} = \begin{pmatrix} t_w a & \delta_s(t_w)a + t_w \delta_s(a) \\ 0 & t_w a \end{pmatrix} - \begin{pmatrix} t_w \delta_w(a)t_w & t_w \delta_w(a)\delta_s(t_w) + t_w \delta_s(\delta_w(a))t_w + \delta_s(t_w)\delta_w(a)t_w \\ 0 & t_w \delta_w(a)t_w \end{pmatrix}.$$

Since $at_w = t_w a - t_w \delta_w(a)t_w$, the matrix equality is equivalent to the equality

$$\delta_s(a)t_w + a\delta_s(t_w) \stackrel{(*)}{=} \delta_s(t_w)a + t_w \delta_s(a) - t_w \delta_w(a)\delta_s(t_w) - t_w \delta_s(\delta_w(a))t_w - \delta_s(t_w)\delta_w(a)t_w.$$

After substituting $-t_w \delta_s(w)t_w$ for $\delta_s(t_w)$, the right hand side of the equality $(*)$ is equal to

$$\begin{aligned} & -t_w \delta_s(w)t_w a + t_w \delta_s(a) + t_w \delta_w(a)t_w \delta_s(w)t_w - t_w \delta_s(\delta_w(a))t_w + t_w \delta_s(w)t_w \delta_w(a)t_w \\ &= -t_w \delta_s(w)(at_w + t_w \delta_w(a)t_w) + t_w \delta_s(a) + t_w \delta_w(a)t_w \delta_s(w)t_w - t_w \delta_s(\delta_w(a))t_w + t_w \delta_s(w)t_w \delta_w(a)t_w \\ &= -t_w \delta_s(w)at_w + t_w \delta_s(a) + t_w \delta_w(a)t_w \delta_s(w)t_w - t_w \delta_s(\delta_w(a))t_w \end{aligned}$$

Now, the left hand side of (*) is

$$\begin{aligned}\delta_s(a)t_w + a\delta_s(t_w) &= t_w\delta_s(a) - t_w\delta_w(\delta_s(a))t_w - at_w\delta_s(w)t_w \\ &= t_w\delta_s(a) - t_w\delta_w(\delta_s(a))t_w - t_wa\delta_s(w)t_w + t_w\delta_w(a)t_w\delta_s(w)t_w\end{aligned}$$

After eliminating equal terms on both sides of (*), we see that it holds if and only if

$$-t_w\delta_s(w)at_w - t_w\delta_s(\delta_w(a))t_w = -t_w\delta_w(\delta_s(a))t_w - t_wa\delta_s(w)t_w.$$

Equivalently,

$$-t_w([w, s], a] + [[a, w], s] + [[s, a], w])t_w = 0. \quad (1.5)$$

By the Jacobi identity, equality (1.5) holds. Therefore $\Phi: S \rightarrow \mathbb{T}_2(S)$ must be a k -algebra homomorphism. This shows the existence of the claimed derivation $\delta_s: R\langle t_w; \delta_w \rangle \rightarrow R\langle t_w; \delta_w \rangle$.

(2) By (1), the map $\Phi: R\langle t_w; \delta_w \rangle \rightarrow \mathbb{T}_2(R\langle t_w; \delta_w \rangle)$, defined by $f \mapsto \begin{pmatrix} f & \delta_s(f) \\ 0 & f \end{pmatrix}$ is a k -algebra homomorphism. Note that $\mathbb{T}_2(R\langle t_w; \delta_w \rangle)$ is canonically isomorphic to $\mathbb{T}_2(R)\langle T_w; \Delta_w \rangle$ where $T_w = \begin{pmatrix} t_w & 0 \\ 0 & t_w \end{pmatrix}$ and $\Delta_w: \mathbb{T}_2(R) \rightarrow \mathbb{T}_2(R)$ is defined by $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} \delta_w(a) & \delta_w(b) \\ 0 & \delta_w(c) \end{pmatrix}$. Note also that $\mathbb{T}_2(R[[t_w; \delta_w]])$ is canonically isomorphic to $\mathbb{T}_2(R)[[T_w; \Delta_w]]$.

One can prove that $\Phi(t_w^n R\langle t_w; \delta_w \rangle) \subseteq T_w^n \mathbb{T}_2(R)\langle T_w; \Delta_w \rangle$ for each positive integer n . These induce a k -algebra homomorphism between the completions $R[[t_w; \delta_w]] \rightarrow \mathbb{T}_2(R[[t_w; \delta_w]])$ defined by $\delta_s(\sum_i t_w^i a_i) = \sum_i \delta_s(t_w^i a_i)$. This morphism extends to $R((t_w; \delta_w))$, as desired. \square

2. THE CASE OF THE HEISENBERG LIE ALGEBRA

The *Heisenberg Lie k -algebra* is the Lie k -algebra with presentation

$$H = \langle x, y \mid [[y, x], x] = [[y, x], y] = 0 \rangle. \quad (2.1)$$

The Heisenberg Lie k -algebra can also be characterized as the unique Lie k -algebra of dimension three such that $[H, H]$ has dimension one and $[H, H]$ is contained in the center of H , see [11, Section 4.III].

Our aim in this section is to prove that $\mathfrak{D}(H)$ contains free algebras generated by symmetric elements with respect to the principal involution.

2.1. A technical result. In this subsection, we prove a result that will be useful to obtain free algebras generated by symmetric elements.

An *ordered group* $(G, <)$ is a group G together with a total order relation $<$ such that the product in G is compatible with $<$. More precisely, the inequality $x < y$ implies that $zx < zy$ and $xz < yz$ for all $x, y, z \in G$.

Lemma 2.1. *Let $(G, <)$ be an ordered group. Let $x, y \in G$ be different positive elements in G (i.e. $1 < x, y$) such that the monoid generated by them is the free monoid on the set $\{x, y\}$. Let k be a field and consider the group algebra $k[G]$. Then $\{x + x^{-1}, y + y^{-1}\}$ generate a free k -algebra on the set $\{x + x^{-1}, y + y^{-1}\}$ inside $k[G]$.*

Proof. Define $X = x + x^{-1}$ and $Y = y + y^{-1}$.

Let M_1, M_2, \dots, M_r be different (monomial) words on two letters and let $a_1, a_2, \dots, a_r \in k \setminus \{0\}$. We have to show that

$$M_1(X, Y)a_1 + \dots + M_r(X, Y)a_r \neq 0.$$

First note that $M_i(x, y) \neq M_j(x, y)$ if $i \neq j$ because the monoid generated by $\{x, y\}$ is free. Thus we may suppose that $M_1(x, y) > M_2(x, y) > \dots > M_r(x, y)$.

Now notice that

$$X^n = (x + x^{-1})^n = x^n + \sum_{-n \leq i < n} x^i \alpha_i, \quad Y^m = (y + y^{-1})^m = y^m + \sum_{-m \leq j < m} y^j \beta_j, \quad (2.2)$$

for some $\alpha_i, \beta_j \in k$.

Recall that in the ordered group G , if $c < d$ and $e < f$, then $ce < df$. This and (2.2) imply that, for fixed $j \in \{1, \dots, r\}$, $M_j(X, Y) = M_j(x, y) + \sum_l N_l b_l$ where $N_l \in G$, $b_l \in k$ and $M_j(x, y) > N_l$ for all l .

Hence $M_1(X, Y)a_1 + \dots + M_r(X, Y)a_r = M_1(x, y)a_1 + \sum_t P_t c_t$ where $P_t \in G$, $c_t \in k$ and $M_1(x, y) > P_t$ for all t . Therefore, $M_1(X, Y)a_1 + \dots + M_r(X, Y)a_r \neq 0$, as desired. \square

Corollary 2.2. *Let G be the free group on the set of two elements $\{x, y\}$. Let k be a field and consider the group algebra $k[G]$. Then the k -algebra generated by $x + x^{-1}$ and $y + y^{-1}$ inside $k[G]$ is free on $\{x + x^{-1}, y + y^{-1}\}$.*

Proof. By Lemma 2.1, it is enough to show that there exists an ordered group structure $(G, <)$ on G such that $1 < x, y$. It is well known that this can be done. We sketch a proof of this fact.

Let $G_1 = G$ and $G_{i+1} = [G, G_i]$ for $i \geq 1$.

We choose a special ordering for the group G_1/G_2 which is free abelian with basis $\{xG_2, yG_2\}$. We can make $x^{n_1}y^{m_1}G_2 > x^{n_2}y^{m_2}G_2$ if either $n_1 > n_2$, or $n_1 = n_2$ and $m_1 > m_2$.

It is known that G_i/G_{i+1} is a free abelian group of finite rank. Thus we can order these groups, for example lexicographically. These orderings induce an ordering in G such that $x > 1$ and $y > 1$, see for example [14, p. 97]. \square

2.2. Cauchon's result and consequences. In this subsection, we present the main result in [2] and a consequence of it that will be useful for our purposes.

Let k be a field. Let $K = k(t)$ be the field of fractions of the polynomial ring $k[t]$ in the variable t . Let σ be a k -automorphism of K . We will consider the skew polynomial ring $K[p; \sigma]$. The elements of $K[p; \sigma]$ are "right polynomials" of the form $\sum_{i=0}^n p^i a_i$, where the coefficients a_i are in K . The multiplication is determined by

$$ap = p\sigma(a) \quad \text{for all } a \in K.$$

It is known that $K[p; \sigma]$ is a noetherian domain and therefore it has an Ore division ring of fractions $D = K(p; \sigma)$.

Since σ is an automorphism of K , $\sigma(t) = \frac{at+b}{ct+d}$ where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(k)$ defines a homography h of the projective line $\Delta = \mathbb{P}_1(k) = k \cup \{\infty\}$, $h: \Delta \rightarrow \Delta$, $z \mapsto h(z) = \frac{az+b}{cz+d}$.

We denote by $\mathcal{H} = \{h^n \mid n \in \mathbb{Z}\}$ the subgroup of the projective linear group $PGL_2(k)$ generated by h . The group \mathcal{H} acts on Δ . If $z \in \Delta$, we denote by $\mathcal{H} \cdot z = \{h^n(z) \mid n \in \mathbb{Z}\}$ the orbit of z under the action of \mathcal{H} .

Cauchon's Theorem. *Let α and β be two elements of k such that the orbits $\mathcal{H} \cdot \alpha$ and $\mathcal{H} \cdot \beta$ are infinite and different. Let s and u be the two elements of D defined by*

$$s = (t - \alpha)(t - \beta)^{-1}, \quad u = (1 - p)(1 + p)^{-1}.$$

If the characteristic of k is different from 2, then the k -subalgebra Ω of D generated by $\xi = s$, $\eta = usu^{-1}$, ξ^{-1} and η^{-1} is the free group k -algebra on the set $\{\xi, \eta\}$. \square

We will need the following consequence of Cauchon's Theorem.

Proposition 2.3. *Let k be a field of characteristic zero and $K = k(t)$ the field of fractions of the polynomial ring $k[t]$. Let $\sigma: K \rightarrow K$ be the automorphism of k -algebras determined by $\sigma(t) = t - 1$. Consider the skew polynomial ring $K[p; \sigma]$ and its Ore division ring of fractions $K(p; \sigma)$. Set*

$$s = \left(t - \frac{5}{6}\right)\left(t - \frac{1}{6}\right)^{-1}, \quad u = (1 - p^2)(1 + p^2)^{-1}.$$

Then the k -subalgebra of $K(p; \sigma)$ generated by $\{s + s^{-1}, u(s + s^{-1})u^{-1}\}$ is the free k -algebra on the set $\{s + s^{-1}, u(s + s^{-1})u^{-1}\}$.

Proof. We will apply Cauchon's Theorem to the skew polynomial ring $K[p^2; \sigma^2]$, where $\sigma^2: K \rightarrow K$ is given by $\sigma^2(t) = t - 2$.

Let $\alpha = \frac{5}{6} \in k$ and $\beta = \frac{1}{6}$. Let \mathcal{H} be defined as above. Consider the orbits $\mathcal{H} \cdot \alpha = \{\frac{5}{6} - 2n \mid n \in \mathbb{Z}\}$, $\mathcal{H} \cdot \beta = \{\frac{1}{6} - 2n \mid n \in \mathbb{Z}\}$ which are infinite and different.

Then, by Cauchon's Theorem, $s = (t - \alpha)(t - \beta)^{-1}$ and $u = (1 - p^2)(1 + p^2)^{-1}$ are such that the k -algebra generated by $\xi = s$, $\eta = usu^{-1}$, ξ^{-1} and η^{-1} is the free group k -algebra on the free generators $\{\xi, \eta\}$.

By Corollary 2.2, the k -algebra generated by $\{s + s^{-1}, u(s + s^{-1})u^{-1}\}$ is the free k -algebra on the set $\{s + s^{-1}, u(s + s^{-1})u^{-1}\}$. \square

2.3. The Heisenberg Lie algebra case. We will work with the skew polynomial ring considered in Section 2.2. More precisely, let $K = k(t)$ be the field of fractions of the polynomial ring $k[t]$. Let $\sigma: K \rightarrow K$ be the k -automorphism determined by $\sigma(t) = t - 1$. Consider the skew polynomial ring $K[p; \sigma]$ and its Ore division ring of fractions $K(p; \sigma)$.

Let now $A_1 = k\langle X, Y \mid YX - XY = 1 \rangle$ be the first Weyl algebra, and let D_1 be its Ore division ring of fractions. The following is well known.

Lemma 2.4. *There exists a k -algebra isomorphism $\psi: D_1 \rightarrow K(p; \sigma)$ such that $\psi(Y) = p$ and $\psi(X) = p^{-1}t$.*

Proof. Since $p(p^{-1}t) - (p^{-1}t)p = t - p^{-1}p(t - 1) = 1$, there exists a morphism of k -algebras $\psi: A_1 \rightarrow K(p; \sigma)$ such that $\psi(Y) = p$ and $\psi(X) = p^{-1}t$. This morphism is injective because A_1 is a simple ring. Moreover, by the universal property of Ore localizations, it can be extended to a morphism of k -algebras $\psi: D_1 \rightarrow K(p; \sigma)$ because ψ was injective and

$K(p; \sigma)$ is a division ring. Clearly ψ is injective for D_1 is a division ring. Now ψ is surjective because $\psi(Y) = p$ and $\psi(YX) = \psi(Y)\psi(X) = pp^{-1}t = t$. \square

Recall the Heisenberg Lie algebra H given by the presentation (2.1) and let $z = [y, x]$. The following is the left version of [16, Lemma 7].

Lemma 2.5. *There exists a morphism of k -algebras $\Upsilon: U(H) \rightarrow D_1$ defined by $\Upsilon(x) = X$, $\Upsilon(y) = Y$ and $\Upsilon(z) = 1$. The morphism Υ has the following properties:*

- (1) *The kernel of Υ is $I = U(H)(z - 1)$, the ideal of $U(H)$ generated by the central element $z - 1$.*
- (2) *The set $\mathfrak{S} = U(H) \setminus I$ is an Ore subset of $U(H)$ and the Ore localization of $U(H)$ at \mathfrak{S} , $\mathfrak{S}^{-1}U(H)$, is a local ring with maximal ideal $\mathfrak{S}^{-1}I$.*

Therefore Υ can be extended to a surjective morphism of k -algebras $\Upsilon: \mathfrak{S}^{-1}U(H) \rightarrow D_1$. \square

Now we are ready to prove the main result of this section. During the proof, we will follow the notation of the foregoing results.

Theorem 2.6. *Let k be a field of characteristic zero. Let H be the Heisenberg Lie k -algebra, i.e.*

$$H = \langle x, y \mid [[y, x], x] = [[y, x], y] = 0 \rangle.$$

Let $U(H)$ be the universal enveloping algebra of H and $\mathfrak{D}(H)$ be the Ore division ring of fractions of $U(H)$. Set $z = [y, x]$, $V = \frac{1}{2}z(xy + yx)z$, and consider the following elements of $\mathfrak{D}(H)$:

$$S = (V - \frac{1}{3}z^3)(V + \frac{1}{3}z^3)^{-1} + (V - \frac{1}{3}z^3)^{-1}(V + \frac{1}{3}z^3),$$

$$T = (z + y^2)^{-1}(z - y^2)S(z + y^2)(z - y^2)^{-1}.$$

The following hold true.

- (1) *The k -subalgebra of $\mathfrak{D}(H)$ generated by S and T is the free k -algebra on the set $\{S, T\}$.*
- (2) *The elements S and T are symmetric with respect to the principal involution on $\mathfrak{D}(H)$.*

Proof. Consider the morphism of rings $\Phi = \psi\Upsilon: \mathfrak{S}^{-1}U(H) \rightarrow K(p; \sigma)$. Recall that in $K(p; \sigma)$, we have and $tp = p(t - 1)$.

First notice that $\Phi(y) = p$, $\Phi(x) = p^{-1}t$ and $\Phi(z) = 1$. Thus $\Phi(V) = \Phi(\frac{1}{2}z(xy + yx)z) = t - \frac{1}{2}$, $\Phi(V - \frac{1}{3}z^3) = t - \frac{5}{6}$, $\Phi(V + \frac{1}{3}z^3) = t - \frac{1}{6}$, $\Phi(z + y^2) = 1 + p^2$, $\Phi(z - y^2) = 1 - p^2$. Hence $V - \frac{1}{3}z^3$, $V + \frac{1}{3}z^3$, $z + y^2$, $z - y^2$ are invertible in $\mathfrak{S}^{-1}U(H)$. Thus $S, T \in \mathfrak{S}^{-1}U(H)$. Moreover, following the notation of Proposition 2.3,

$$\Phi(S) = (t - \frac{5}{6})(t - \frac{1}{6})^{-1} + (t - \frac{1}{6})(t - \frac{5}{6})^{-1} = s + s^{-1},$$

$$\Phi((z + y^2)^{-1}(z - y^2)) = (1 + p^2)^{-1}(1 - p^2) = (1 - p^2)(1 + p^2)^{-1} = u,$$

$$\Phi((z + y^2)(z - y^2)^{-1}) = (1 + p^2)(1 - p^2)^{-1} = u^{-1}.$$

Hence $\Phi(T) = u(s + s^{-1})u^{-1}$.

By Proposition 2.3, the set $\{s + s^{-1}, u(s + s^{-1})u^{-1}\}$ generates a free k -algebra. Therefore, the k -algebra generated by S and T is a free k -algebra on the set $\{S, T\}$.

It remains to show that S and T are symmetric with respect to the principal involution $*$.

Clearly $V^* = V$. Thus $(V - \frac{1}{3}z^3)^* = (V + \frac{1}{3}z^3)$ and $(V + \frac{1}{3}z^3)^* = V - \frac{1}{3}z^3$. Hence

$$S^* = \left((V - \frac{1}{3}z^3)(V + \frac{1}{3}z^3)^{-1} + (V - \frac{1}{3}z^3)^{-1}(V + \frac{1}{3}z^3) \right)^* = S$$

Now

$$\begin{aligned} ((z + y^2)^{-1}(z - y^2))^* &= (z - y^2)^* ((z + y^2)^*)^{-1} \\ &= (-z - y^2)(-z + y^2)^{-1} \\ &= (z + y^2)(z - y^2)^{-1}. \end{aligned}$$

Thus

$$\begin{aligned} T^* &= ((z + y^2)^{-1}(z - y^2)S(z + y^2)(z - y^2)^{-1})^* \\ &= ((z + y^2)(z - y^2)^{-1})^* S^* ((z + y^2)^{-1}(z - y^2))^* \\ &= (z + y^2)^{-1}(z - y^2)S(z + y^2)(z - y^2)^{-1} \\ &= T. \end{aligned}$$

Therefore S and T are symmetric with respect to the principal involution. \square

Simpler elements S and T , symmetric with respect to the principal involution and that generate a free algebra in $\mathfrak{D}(H)$, can be found. The reason why we decided to pick these more complicated elements is a technical one. It should become clearer at the end of the proof of Theorem 4.2.

3. THE CASE OF A RESIDUALLY NILPOTENT LIE ALGEBRA

In this section, we develop a technique to obtain free algebras in $\mathcal{D}(L)$, where L is a (generalization of) a residually nilpotent Lie algebra from the ones obtained in $\mathfrak{D}(H)$, where H is the Heisenberg Lie algebra.

Let L a Lie k -algebra generated by two elements u, v . Let $H = \langle x, y \mid [[y, x], x] = [[y, x], y] = 0 \rangle$ be the Heisenberg Lie k -algebra. Suppose that there exists a Lie k -algebra homomorphism

$$L \xrightarrow{\rho} H, \quad u \mapsto x, \quad v \mapsto y.$$

Define $w = [v, u]$ and $z = [y, x]$. Let $N = \ker \rho$. Thus N is a (Lie) ideal of L .

By the universal property of universal enveloping algebras, ρ can be uniquely extended to a k -algebra homomorphism $\psi: U(L) \rightarrow U(H)$ between the corresponding universal enveloping algebras. Note that $\ker \psi$ is the ideal of $U(L)$ generated by N . The restriction $\psi|_{U(N)}$ coincides with the augmentation map $\varepsilon: U(N) \rightarrow k$.

By the PBW-Theorem, the elements of $U(H)$ are uniquely expressed as finite sums $\sum_{l,m,n \geq 0} x^l y^m z^n a_{lmn}$ with $a_{lmn} \in k$. Let δ_x be the inner k -derivation of $U(H)$ determined by x , i.e. $\delta_x(f) = [f, x] = fx - xf$ for all $f \in U(H)$. Notice that the k -subalgebra of $U(H)$ generated by z can be regarded as $k[z]$, the polynomial algebra in the variable z .

Since $[z, y] = 0$, the k -subalgebra of $U(H)$ generated by $\{y, z\}$ can be viewed as $k[z][y]$. As $\delta_x(z) = [z, x] = 0$, $\delta_x(y) = [y, x] = z \in k[z][y]$, one can prove that

$$U(H) = k[z][y][x; \delta_x] \quad (3.1)$$

because (a), (b), (c), (d) in Section 1.2 are easily verified.

Consider now $U(L)$, the universal enveloping algebra of L . By the PBW-Theorem, the elements of $U(L)$ can be uniquely expressed as finite sums $\sum_{l,m,n \geq 0} u^l v^m w^n f_{lmn}$ with $f_{lmn} \in U(N)$. Since N is an ideal of L , the inner derivations $\delta_u, \delta_v, \delta_w$ of $U(L)$ defined by u, v, w , respectively, are such that $\delta_u(U(N)) \subseteq U(N)$, $\delta_v(U(N)) \subseteq U(N)$, $\delta_w(U(N)) \subseteq U(N)$. The k -subalgebra of $U(L)$ generated by $U(N)$ and w is $U(N)[w; \delta_w]$ for properties (a), (b), (c), (d) in Section 1.2 are easily verified. Since $\delta_v(w) \in U(N) \subseteq U(N)[w; \delta_w]$, one can prove, as before, that the k -subalgebra of $U(L)$ generated by $U(N)$ and $\{w, v\}$ is $U(N)[w; \delta_w][v; \delta_v]$. Furthermore, since $\delta_u(v) = w$ and $\delta_u(w) \in U(N)$,

$$U(L) = U(N)[w; \delta_w][v; \delta_v][u; \delta_u]. \quad (3.2)$$

By (3.1), $U(H) = k[z][y][x; \delta_x]$. By applying Lemma 1.4(2) twice, we get

$$U(H) \hookrightarrow k((t_z))((t_y))((t_x; \delta_x)). \quad (3.3)$$

More precisely, the first time, we obtain that δ_x can be extended to $k((t_z))$. Thus obtaining the embedding $U(H) \hookrightarrow k((t_z))[y][x; \delta_x]$. Applying Lemma 1.4 a second time with the role of R played by $k((t_x))[y]$ and the role of w by y , we get that δ_x can be extended to $k((t_z))((t_y))$. Hence we obtain the embedding (3.3).

By (3.2), $U(L) = U(N)[w; \delta_w][v; \delta_v][u; \delta_u]$. Again, Lemma 1.4(2) yields an embedding

$$U(L) \hookrightarrow U(N)((t_w; \delta_w))((t_v; \delta_v))((t_u; \delta_u)). \quad (3.4)$$

More precisely, applying Lemma 1.4(2) twice, we get extensions of δ_v and δ_u to $U(N)((t_w; \delta_w))$, resulting in an embedding $U(L) \hookrightarrow U(L)((t_w; \delta_w))[t_v; \delta_v][t_u; \delta_u]$. Again, by Lemma 1.4(2), we get an extension of δ_u to $U(L)((t_w; \delta_w))((t_v; \delta_v))$ and, hence, an embedding of $U(L)$ into $U(N)((t_w; \delta_w))((t_v; \delta_v))[t_u; \delta_u]$, and from this we get (3.4).

In this setting, we prove the next results.

Lemma 3.1. *There exists a commutative diagram of embeddings of k -algebras*

$$\begin{array}{ccc} U(L) = U(N)[w; \delta_w][v; \delta_v][u; \delta_u] & \hookrightarrow & \mathfrak{D}(N)[w; \delta_w][v; \delta_v][u; \delta_u] \\ \downarrow & \searrow & \swarrow \downarrow \\ & \mathfrak{D}(L) & \\ \downarrow & \searrow & \swarrow \downarrow \\ U(N)((t_w; \delta_w))((t_v; \delta_v))((t_u; \delta_u)) & \hookrightarrow & \mathfrak{D}(N)((t_w; \delta_w))((t_v; \delta_v))((t_u; \delta_u)) \end{array}$$

Proof. By Proposition 1.2, the division k -subalgebra of $\mathfrak{D}(L)$ generated by $U(N)$ is $\mathfrak{D}(N)$ and the standard monomials on the set $\{u, v, w\}$ are $\mathfrak{D}(N)$ -linearly independent.

Recall that if f is an invertible element in a k -algebra and δ is a derivation, then $\delta(f^{-1}) = -f^{-1}\delta(f)f^{-1}$. Thus, since the image of $U(N)$ under the derivations $\delta_u, \delta_v, \delta_w$ is contained

in $U(N)$, one can prove that $\delta_u(\mathfrak{D}(N)) \subseteq \mathfrak{D}(N)$, $\delta_v(\mathfrak{D}(N)) \subseteq \mathfrak{D}(N)$ and $\delta_w(\mathfrak{D}(N)) \subseteq \mathfrak{D}(N)$. Hence we obtain the embeddings

$$\begin{array}{ccc} U(L) = U(N)[w; \delta_w][v; \delta_v][u; \delta_u] & \hookrightarrow & \mathfrak{D}(N)[w; \delta_w][v; \delta_v][u; \delta_u] \\ \downarrow & \searrow & \swarrow \\ & \mathfrak{D}(L) & \\ \downarrow & & \downarrow \\ U(N)((t_w; \delta_w))((t_v; \delta_v))((t_u; \delta_u)) & \hookrightarrow & \mathfrak{D}(N)((t_w; \delta_w))((t_v; \delta_v))((t_u; \delta_u)) \end{array}$$

The remaining embedding $\mathfrak{D}(L) \hookrightarrow \mathfrak{D}(N)((t_w; \delta_w))((t_v; \delta_v))((t_u; \delta_u))$ is obtained as follows. The division k -algebra $\mathfrak{D}(L)$ is generated, as a division k -algebra, by $U(L)$. Hence $\mathfrak{D}(L)$ is generated by $\mathfrak{D}(N)[u; \delta_u][v; \delta_v][w; \delta_w]$. The k -algebra $\mathfrak{D}(N)[w; \delta_w][v; \delta_v][u; \delta_u]$ is an Ore domain (see for example [13, Theorem 10.28]). Therefore $\mathfrak{D}(L)$ is the Ore division ring of fractions of $\mathfrak{D}(N)[w; \delta_w][v; \delta_v][u; \delta_u]$. The series ring $\mathfrak{D}(N)((t_w; \delta_w))((t_v; \delta_v))((t_u; \delta_u))$ is a division k -algebra that contains $\mathfrak{D}(N)[w; \delta_w][v; \delta_v][u; \delta_u]$; thus it contains $\mathfrak{D}(L)$, as desired. \square

Lemma 3.2. *Let $\varepsilon: U(N) \rightarrow k$ denote the augmentation map. The following hold true.*

(1) *There exists a k -algebra homomorphism*

$$\Phi_w: U(N)((t_w; \delta_w)) \rightarrow k((t_z)), \quad \sum_i t_w^i f_i \mapsto \sum_i t_z^i \varepsilon(f_i),$$

where $f_i \in U(N)$ for each i .

(2) *There exists a k -algebra homomorphism*

$$\Phi_v: U(N)((t_w; \delta_w))((t_v; \delta_v)) \rightarrow k((t_z))((t_y)), \quad \sum_i t_v^i g_i \mapsto \sum_i t_y^i \Phi_w(g_i),$$

where $g_i \in U(N)((t_w; \delta_w))$ for each i .

(3) *There exists a k -algebra homomorphism*

$$\Phi_u: U(N)((t_w; \delta_w))((t_v; \delta_v))((t_u; \delta_u)) \rightarrow k((t_z))((t_y))((t_x; \delta_x)), \quad \sum_i t_u^i h_i \mapsto \sum_i t_x^i \Phi_v(h_i),$$

where $h_i \in U(N)((t_w; \delta_w))((t_v; \delta_v))$ for each i .

Proof. Recall that $\ker \varepsilon$ is the ideal of $U(N)$ generated by N . Also, any element $f \in U(N)$ is of the form $f = a + b$ with $a \in k$ and $b \in \ker \varepsilon$. Since N is an ideal of H , $\delta_s(n) = [n, s] \in N$ for any $n \in N$, $s \in L$. Thus

$$\delta_s(U(N)) \subseteq \ker \varepsilon, \quad \text{for any } s \in L. \quad (3.5)$$

(1) The polynomial ring $k[z]$ can be regarded as $k[z; \delta_z]$ where $\delta_z = 0$. By Lemma 1.3, it is enough to prove that $\varepsilon(\delta_w(f)) = \delta_z(\varepsilon(f)) = 0$ for all $f \in U(N)$. This holds by (3.5).

(2) The polynomial ring $k((t_z))[y]$ can be viewed as $k((t_z))[y; \delta_y]$ where $\delta_y = 0$. By Lemma 1.3, it is enough to prove that

$$\Phi_w(\delta_v(g)) = \delta_y(\Phi_w(g)) = 0 \quad \text{for all } g \in U(N)((t_w; \delta_w)). \quad (3.6)$$

Observe that it is enough to prove this equality for $g \in U(N)[[t_w; \delta_w]]$. Let $g = \sum_{i \geq 0} t_w^i f_i$ with $f_i \in U(N)$ for all i . By Lemma 1.4(2),

$$\delta_v \left(\sum_{i \geq 0} t_w^i f_i \right) = \sum_{i \geq 0} \delta_v(t_w^i f_i) = \sum_{i \geq 0} \left(t_w^i \delta_v(f_i) + \delta_v(t_w^i) f_i \right).$$

By (3.5), $\delta_v(f_i) \in \ker \varepsilon$. Therefore

$$\Phi_w \left(\sum_{i \geq 0} t_w^i \delta_v(f_i) \right) = \sum_{i \geq 0} t_w^i \varepsilon(\delta_v(f_i)) = 0.$$

By Lemma 1.4(2),

$$\delta_v(t_w) = -t_w \delta_v(w) t_w = \sum_{i=1}^{\infty} t_w^{i+1} \delta_w^{i-1}(\delta_v(w)) (-1)^i.$$

Therefore $\delta_v(t_w)$ is a series with coefficients in $\ker \varepsilon$. Suppose that $\delta_v(t_w^{i-1})$ is a series with coefficients in $\ker \varepsilon$. Then

$$\delta_v(t_w^i) = \delta_v(t_w^{i-1} t_w) = t_w^{i-1} \delta_v(t_w) + \delta_v(t_w^{i-1}) t_w.$$

From this, it is not difficult to show that $\delta_v(t_w^i)$ is a series with coefficients in $\ker \varepsilon$. Hence $\sum_{i \geq 0} \delta_v(t_w^i) f_i$ is a series with coefficients in $\ker \varepsilon$, and $\Phi_w \left(\sum_{i \geq 0} \delta_v(t_w^i) f_i \right) = 0$. Therefore $\Phi_w(\delta_w(g)) = 0$, as desired.

(3) By Lemma 1.3, it is enough to prove that

$$\Phi_v(\delta_u(h)) = \delta_x(\Phi_v(h)) \text{ for all } h \in U(N)((t_w; \delta_w))((t_v; \delta_v)).$$

Observe that it is enough to prove this equality for $h \in U(N)((t_w; \delta_w))[[t_v; \delta_v]]$. Let $h = \sum_{i \geq 0} t_v^i g_i$ where $g_i \in U(N)((t_w; \delta_w))$ for all i . By Lemma 1.4(2),

$$\delta_u \left(\sum_{i \geq 0} t_v^i g_i \right) = \sum_{i \geq 0} \left(t_v^i \delta_u(g_i) + \delta_u(t_v^i) g_i \right).$$

In (3.6), we proved that $\delta_v(g_i) \in \ker \Phi_w$. In the same way, one can show that $\delta_u(g_i) \in \ker \Phi_w$. Thus $\Phi_v \left(\sum_{i \geq 0} t_v^i \delta_u(g_i) \right) = \sum_{i \geq 0} t_v^i \Phi_w(\delta_u(g_i)) = 0$. Hence we only have to worry about the term $\sum_{i \geq 0} \delta_u(t_v^i) g_i$. We prove, by induction on i , that $\delta_u(t_v^i) = t_v^{i+1} w(-i) + A_i$, with $A_i \in \ker \Phi_v$. For $i = 0$, the result is clear. For $i = 1$, $\delta_u(t_v) = -t_v \delta_u(v) t_v = t_v w t_v(-1)$. By Lemma 1.4(2),

$$w t_v = \sum_{i \geq 1} t_v^i \delta_v^{i-1}(w) (-1)^{i-1} = t_v w + A'_1,$$

where $A'_1 = \sum_{i \geq 2} t_v^i \delta_v^{i-1}(w) (-1)^{i-1}$, a series in t_v where the coefficient of t_v^i is $\delta_v^{i-1}(w) (-1)^{i-1} \in \ker \varepsilon$ for $i > 1$. Thus $\delta_u(t_v) = t_v^2 w(-1) + t_v A'_1(-1)$. If we set $A_1 = t_v A'_1(-1)$, the result is proved for $i = 1$. Suppose the result holds for $l < i$. Then

$$\begin{aligned} \delta_u(t_v^i) &= \delta_u(t_v^{i-1} t_v) \\ &= t_v^{i-1} \delta_u(t_v) + \delta_u(t_v^{i-1}) t_v \\ &= t_v^{i-1} (t_v^2 w(-1) + A_1) + (t_v^i w(-(i-1)) + A_{i-1}) t_v \\ &= t_v^{i+1} w(-1) + t_v^{i+1} w(-(i-1)) + t_v^{i-1} A_1 + A_{i-1} t_v + t_v^i A'_1(-(i-1)) \\ &= t_v^{i+1} w(-i) + t_v^{i-1} A_1 + A_{i-1} t_v + t_v^i A'_1(-(i-1)). \end{aligned}$$

Now it is not difficult to prove that $A_i = t_v^{i-1}A_1 + A_{i-1}t_v + t_v^iA_1'(-(i-1))$ is a series in t_v with coefficients in $\ker \varepsilon$. Thus $A_i \in \ker \Phi_v$.

Hence, we have, on the one hand,

$$\begin{aligned}
\Phi_v\left(\delta_u\left(\sum_{i \geq 0} t_v^i g_i\right)\right) &= \Phi_v\left(\sum_{i \geq 0} \left(t_v^i \delta_u(g_i) + \delta_u(t_v^i) g_i\right)\right) \\
&= \Phi_v\left(\sum_{i \geq 0} t_v^i \delta_u(g_i)\right) + \Phi_v\left(\sum_{i \geq 0} \delta_u(t_v^i) g_i\right) \\
&= \sum_{i \geq 0} t_y^i \Phi_w(\delta_u(g_i)) + \sum_{i \geq 0} \Phi_v(t_v^{i+1} w(-i) + A_i) \Phi_w(g_i) \\
&= \sum_{i \geq 0} t_y^{i+1} \Phi_w(w(-i)) \Phi_w(g_i) \\
&= \sum_{i \geq 0} t_y^{i+1} z \Phi_w(g_i)(-i).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\delta_x\left(\Phi_v\left(\sum_{i \geq 0} t_v^i g_i\right)\right) &= \delta_x\left(\sum_{i \geq 0} t_y^i \Phi_w(g_i)\right) \\
&= \sum_{i \geq 0} \left(t_y^i \delta_x(\Phi_w(g_i)) + \delta_x(t_y^i) \Phi_w(g_i)\right).
\end{aligned}$$

First observe that $\delta_x(\Phi_w(g_i)) = 0$, because $\Phi_w(g_i)$ is a series in t_z with coefficients in k , and z commutes with x . If we prove, by induction on i , that $\delta_x(t_y^i) = t_y^{i+1}z(-i)$, the result is proved. For $i = 0$, this is clear. For $i = 1$,

$$\delta_x(t_y) = -t_y \delta_x(y) t_y = -t_y z t_y = t_y^2 z(-1),$$

where we have used that z and y commute (and hence z and t_y). Suppose now that the identity holds for $l < i$. Then

$$\begin{aligned}
\delta_x(t_y^i) &= \delta_x(t_y^{i-1} t_y) \\
&= t_y^{i-1} \delta_x(t_y) + \delta_x(t_y^{i-1}) t_y \\
&= t_y^{i-1} t_y^2 z(-1) + (t_y^i z(-(i-1))) t_y \\
&= t_y^{i+1} z(-1) + t_y^{i+1} z(-(i-1)) = t_y^{i+1} z(-i),
\end{aligned}$$

as desired. \square

Now we are ready to prove the main result of this section.

Theorem 3.3. *Let k be a field of characteristic zero, let $H = \langle x, y \mid [[y, x], x] = [[y, x], y] = 0 \rangle$ be the Heisenberg Lie k -algebra and let L be a Lie k -algebra generated by two elements u, v . Suppose that there exists a Lie k -algebra homomorphism*

$$L \rightarrow H, \quad u \mapsto x, \quad v \mapsto y.$$

Let $w = [v, u]$, $V = \frac{1}{2}w(uv + vu)w$, and consider the following elements of $\mathfrak{D}(L)$:

$$S = (V - \frac{1}{3}w^3)(V + \frac{1}{3}w^3)^{-1} + (V - \frac{1}{3}w^3)^{-1}(V + \frac{1}{3}w^3),$$

$$T = (w + v^2)^{-1}(w - v^2)S(w + v^2)(w - v^2)^{-1}.$$

The following hold true.

- (1) The k -subalgebra of $\mathfrak{D}(L)$ generated by S and T is the free k -algebra on the set $\{S, T\}$.
- (2) The elements S and T are symmetric with respect to the principal involution on $\mathfrak{D}(L)$.

Proof. Define $z = [y, x] \in H$. Consider the embedding $U(H) \hookrightarrow k((t_z))((t_y))((t_x; \delta_x))$ given in (3.3). Since $k((t_z))((t_y))((t_x; \delta_x))$ is a division k -algebra and $U(H)$ is an Ore domain, it extends to an embedding $\mathfrak{D}(H) \hookrightarrow k((t_z))((t_y))((t_x; \delta_x))$.

Let $N = \ker(L \rightarrow H)$ and consider the embedding $U(L) \hookrightarrow U(N)((t_w; \delta_w))((t_v; \delta_v))((t_u; \delta_u))$ given in (3.4). Let $\Phi_u: U(N)((t_w; \delta_w))((t_v; \delta_v))((t_u; \delta_u)) \rightarrow k((t_z))((t_y))((t_x; \delta_x))$ be the homomorphism given in Lemma 3.2.

Define the following elements in $\mathfrak{D}(H)$: $V_H = \frac{1}{2}z(xy + yx)z$,

$$S_H = (V_H - \frac{1}{3}z^3)(V_H + \frac{1}{3}z^3)^{-1} + (V_H - \frac{1}{3}z^3)^{-1}(V_H + \frac{1}{3}z^3),$$

$$T_H = (z + y^2)^{-1}(z - y^2)S_H(z + y^2)(z - y^2)^{-1}.$$

By Theorem 2.6, the k -algebra generated by S_H and T_H is the free k -algebra on the set $\{S_H, T_H\}$.

We claim that the elements $V - \frac{1}{3}w^3$, $V + \frac{1}{3}w^3$, $w + v^2$ and $w - v^2$ are all invertible in $U(N)((t_w; \delta_w))((t_v; \delta_v))((t_u; \delta_u))$. If the claim is true, we will have $\Phi_u(V) = V_H$, $\Phi_u(S) = S_H$ and $\Phi_u(T) = T_H$. Moreover, by Lemma 3.1, V , S and T belong to $\mathfrak{D}(L)$. Therefore, the elements S and T are nonzero and invertible in $\mathfrak{D}(L)$, and the k -algebra generated by them is the free k -algebra on the set $\{S, T\}$.

We proceed to prove the claim. We begin with the element $w + v^2 = t_w^{-1} + t_v^{-2}$. As a series in t_v , this element is invertible in $U(N)((t_w; \delta_w))((t_v; \delta_v))$ if and only if the coefficient of t_v^{-2} is invertible in the ring of coefficients $U(N)((t_w; \delta_w))$. The coefficient is 1, which is clearly invertible. Similarly, it can be proved that $w - v^2$ is invertible. Now we show that $V + \frac{1}{3}w^3$ is invertible in $U(N)((t_w; \delta_w))((t_v; \delta_v))((t_u; \delta_u))$. First we obtain an expression of $V + \frac{1}{3}w^3$ as a series in t_u .

$$\begin{aligned} V + \frac{1}{3}w^3 &= \frac{1}{2}w(uv + vu)w + \frac{1}{3}w^3 \\ &= \frac{1}{2}w(uv + [v, u] + uv)w + \frac{1}{3}w^3 \\ &= \frac{1}{2}w^3 + wuvw + \frac{1}{3}w^3 \\ &= \frac{5}{6}w^3 + wuvw \\ &= \frac{5}{6}t_w^{-3} + wuvw. \end{aligned}$$

Now,

$$\begin{aligned}
wvuw &= (uw + [w, u])vw \\
&= u(vw + [w, v])w + (v[w, u] + [[w, u], v])w \\
&= uvw^2 + u(w[w, v] + [[w, v], w]) + v(w[w, u] + [[w, u], w]) + w[[w, u], v] + [[[w, u], v], w] \\
&= t_u^{-1}t_v^{-1}t_w^{-2} + t_u^{-1}t_w^{-1}[w, v] + t_u^{-1}[[w, v], w] + \\
&\quad t_v^{-1}t_w^{-1}[w, u] + t_v^{-1}[[w, u], w] + t_w^{-1}[[w, u], v] + [[[w, u], v], w] \\
&= t_u^{-1}(t_v^{-1}t_w^{-2} + t_w^{-1}[w, v] + [[w, v], w]) + \\
&\quad t_v^{-1}t_w^{-1}[w, u] + t_v^{-1}[[w, u], w] + t_w^{-1}[[w, u], v] + [[[w, u], v], w].
\end{aligned}$$

Thus, as a series in t_u , the coefficient of the least element in the support of $V + \frac{1}{3}w^3$ is $t_v^{-1}t_w^{-2} + t_w^{-1}[w, v] + [[w, v], w] \in U(N)((t_w; \delta_w))((t_v; \delta_v))$. Hence $V + \frac{1}{3}w^3$ is invertible in $U(N)((t_w; \delta_w))((t_v; \delta_v))((t_u; \delta_u))$ if and only if $t_v^{-1}t_w^{-2} + t_w^{-1}[w, v] + [[w, v], w]$ is invertible in $U(N)((t_w; \delta_w))((t_v; \delta_v))$. But, as a series in t_v with coefficients in $U(N)((t_w; \delta_w))$, the coefficient of the least element in the support of $t_v^{-1}t_w^{-2} + t_w^{-1}[w, v] + [[w, v], w]$ is t_w^{-2} . Clearly, t_w^{-2} is invertible in $U(N)((t_w; \delta_w))$. This implies that $t_v^{-1}t_w^{-2} + t_w^{-1}[w, v] + [[w, v], w]$ is invertible in $U(N)((t_w; \delta_w))((t_v; \delta_v))$. Therefore $V + \frac{1}{3}w^3$ is invertible in $U(N)((t_w; \delta_w))((t_v; \delta_v))((t_u; \delta_u))$. The case of $V - \frac{1}{3}w^3$ is shown analogously, and the claim is proved.

We have just proved (1). To prove that S and T are symmetric with respect to the principal involution, one proceeds as in the proof of Theorem 2.6 substituting u, v, w for x, y, z , respectively. \square

Let K be a Lie k -algebra. By a *central series* of K we mean a set \mathcal{K} of ideals of K , called the *terms* of K , which is linearly ordered by inclusion, contains 0 and K , and \mathcal{K} also satisfies the following conditions:

- (i) If $0 \neq x \in K$, there are terms of \mathcal{K} which do not contain x and the union of all such terms is a term V_x of \mathcal{K} .
- (ii) If $0 \neq x \in K$, there are terms of \mathcal{K} which contain x and the intersection of all such terms is a term Λ_x of \mathcal{K} .
- (iii) Each factor Λ_x/V_x is central in K , that is, $[K, \Lambda_x] \subseteq V_x$.

Example 3.4. Let K be a Lie k -algebra.

- (a) Suppose that K is nilpotent, that is, there exists a finite sequence of ideals of K

$$K = K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n = 0,$$

such that $[K, K_i] \subseteq K_{i+1}$. Then $\mathcal{K} = \{K_i\}_{i=1}^n$ is a central series of K .

- (b) More generally, suppose that K is a *residually nilpotent* Lie k -algebra, that is, there exists a sequence $\{K_i\}_{i=1}^\infty$ of ideals of K such that $K_1 = K$, $K_i \supseteq K_{i+1}$, $\bigcap_{i=1}^\infty K_i = 0$ and $[K, K_i] \subseteq K_{i+1}$. Then $\mathcal{K} = \{0\} \cup \{K_i\}_{i=1}^\infty$ is a central series of K .
- (c) Even more generally, suppose that K is a *hypo-central* Lie algebra, that is, K is a Lie k -algebra with a sequence of ideals $\{K_\lambda\}_{\lambda < \varepsilon}$, where ε is an ordinal, such that $K_1 = K$, $K_\lambda \supseteq K_{\lambda+1}$, $[K, K_\lambda] \subseteq K_{\lambda+1}$, $K_\lambda = \bigcap_{\beta < \lambda} K_\beta$ when λ is a limit ordinal, and $\bigcap_{\lambda < \varepsilon} K_\lambda = 0$. Then $\mathcal{K} = \{0\} \cup \{K_\lambda\}_{\lambda < \varepsilon}$ is a central series.

(d) Let M be a semigroup with an order relation such that, for all $x, y, z \in M$, we have

(i) $x < y$ implies $zx < zy$ and $xz < yz$, and

(ii) $x < xy$, $x < yx$.

Suppose that a Lie k -algebra K has a family of k -vector subspaces $\{K^x\}_{x \in M}$ such that $K = \bigoplus_{x \in M} K^x$ and $[K^x, K^y] \subseteq K^{xy}$. Define ideals of K by $V_x = \bigcup_{y \geq x} K^y$ and $\Lambda_x = \bigcup_{y > x} K^y$ (it may happen that $V_x = \Lambda_y$ for some x, y). Then $\mathcal{K} = \{V_x, \Lambda_x \mid x \in M\} \cup K \cup \{0\}$ (possibly not disjoint) is a central series.

(e) A Lie k -algebra K is *hypercentral* if there exists a chain of ideals $\{K_\mu\}_{\mu \leq \nu}$ of K (indexed by some ordinal ν) that satisfies the following conditions: $K_0 = 0$, $K_\nu = K$, $K_\mu \subseteq K_{\mu+1}$ for all $0 \leq \mu < \nu$, $K_{\mu'} = \bigcup_{\mu < \mu'} K_\mu$ for all limit ordinals $\mu' \leq \nu$, $[K, K_{\mu+1}] \subseteq K_\mu$ for all $\mu < \nu$, or equivalently, $K_{\mu+1}/K_\mu$ is contained in the center of K/K_μ . Then $\mathcal{K} = \{K_\mu\}_{\mu \leq \nu}$ is a central series.

(f) There exists a concept of orderable Lie algebra, defined in [12]. It is proved in [12, Corollary 3.5] that every orderable Lie k -algebra admits a central series. \square

Corollary 3.5. *Let k be a field of characteristic zero. Let K be a Lie k -algebra with a central series \mathcal{K} .*

Let $u, v \in K$ be such that $[v, u] \neq 0$. Denote by L the Lie k -subalgebra of K generated by $\{u, v\}$. Then there exists an ideal N of L such that L/N is isomorphic to H , the Heisenberg Lie k -algebra. Therefore, if we define $w = [v, u]$, $V = \frac{1}{2}w(uv + vu)w$,

$$S = (V - \frac{1}{3}w^3)(V + \frac{1}{3}w^3)^{-1} + (V - \frac{1}{3}w^3)^{-1}(V + \frac{1}{3}w^3),$$

$$T = (w + v^2)^{-1}(w - v^2)S(w + v^2)(w - v^2)^{-1},$$

the following hold true.

- (1) *The k -subalgebra of $\mathfrak{D}(K)$ generated by S and T is the free k -algebra on the set $\{S, T\}$.*
- (2) *The elements S and T are symmetric with respect to the principal involution on $\mathfrak{D}(K)$.*

Proof. For $x \in L$, let V_x be the union of all terms of \mathcal{K} that do not contain x , and let Λ_x be the intersection of all the terms of \mathcal{K} which contain x .

Notice that $\Lambda_u \subseteq \Lambda_v$ or $\Lambda_v \subseteq \Lambda_u$. Let $t \in \{u, v\}$ be such that $\Lambda_t = \Lambda_u \cap \Lambda_v$. Consider Λ_t and V_t .

Since $[K, \Lambda_t] \subseteq V_t$, the element $w = [v, u] \in V_t$. By definition, $w \in \Lambda_w \setminus V_w$. Since $\Lambda_w \subseteq V_t$, then $u, v \notin \Lambda_w$. Define $N = V_w \cap L$. Since $[v, u] = w \notin V_w$, then $w \notin N$, and thus L/N is not commutative. Observe that $[L, L] \subseteq \Lambda_w$. Therefore $[L, [L, L]] \subseteq [L, L \cap \Lambda_w] \subseteq L \cap V_w = N$. Hence L/N is a noncommutative 3-dimensional Lie k -algebra with basis $\{\bar{u}, \bar{v}, \bar{w}\}$, the classes of u, v and w in L/N . Moreover $[L/N, L/N] = k\bar{w}$ which is contained in the center of L/N . Therefore L/N is the Heisenberg Lie k -algebra.

By Theorem 3.3, the result holds for $\mathfrak{D}(L)$. Since $\mathfrak{D}(L) \hookrightarrow \mathfrak{D}(K)$ by Proposition 1.2(1), the result follows. \square

The first part of proof of Corollary 3.5 consists on showing that there exists an ideal N of L such that L/N is isomorphic to the Heisenberg Lie k -algebra. This argument is analogous to the one used for ordered groups in [24, after Proposition 3.4].

4. THE UNIVERSAL ENVELOPING ALGEBRA IS ORE

We begin this section gathering technical results from [16].

Proposition 4.1. *Let R be an Ore domain with a valuation $\chi: R \rightarrow \mathbb{Z} \cup \{\infty\}$. This valuation induces a filtration*

$$\cdots \supseteq R_{-1} \supseteq R_0 \supseteq R_1 \supseteq \cdots$$

with $R_i = \{a \in R \mid \chi(a) \geq i\}$. Let $R[t, t^{-1}]$ be the Laurent polynomial ring with coefficients in R .

(1) The valuation χ can be extended to $\chi: R[t, t^{-1}] \rightarrow \mathbb{Z} \cup \{0\}$ by defining

$$\chi\left(\sum_{i=1}^n t^i a_i\right) = \min_i \{\chi(a_i) + i\},$$

for all $\sum_{i=1}^n t^i a_i \in R[t, t^{-1}]$.

(2) Let $T = \{x \in R[t, t^{-1}] \mid \chi(x) \geq 0\}$ and $T_0 = \{x \in R[t, t^{-1}] \mid \chi(x) > 0\}$. Then there exists a ring isomorphism

$$\begin{aligned} \text{gr}(R) &\xrightarrow{\varphi} T/T_0 \\ a \in R_i/R_{i+1} &\mapsto t^{-i}a + T_0 \end{aligned}$$

(3) T and $\text{gr}(R)$ are Ore domains.

(4) $\mathcal{S} = T \setminus T_0$ is a left denominator set, $\mathcal{S}^{-1}T$ is a local ring with maximal ideal $\mathcal{S}^{-1}T_0$.

(5) Let D and Δ be the Ore division rings of fractions of T and $\text{gr}(R)$, respectively. The ring homomorphism

$$T \rightarrow T/T_0 \xrightarrow{\varphi^{-1}} \text{gr}(R) \hookrightarrow \Delta$$

can be extended to a ring homomorphism $\psi: \mathcal{S}^{-1}T \rightarrow \Delta$ which induces an isomorphism

$$\mathcal{S}^{-1}T/\mathcal{S}^{-1}T_0 \cong \Delta.$$

This isomorphism extends $T/T_0 \xrightarrow{\varphi^{-1}} \text{gr}(R)$.

Proof. (1) is proved in [4, p. 87].

(2) is shown in [4, p. 87].

(3) is proved in [16, Propositions 17 and 18(i)].

(4) is shown in [16, Proposition 17].

(5) is proved in [16, Proposition 18(ii)]. □

Now we are ready to prove the main result of this section. The technique used is from [16].

Theorem 4.2. *Let k be a field of characteristic zero. Let L be a Lie k -algebra such that its universal enveloping algebra $U(L)$ is an Ore domain. Let $\mathfrak{D}(L)$ be its Ore division ring of fractions. Let $u, v \in L$ such that the Lie subalgebra generated by them is of dimension at least three.*

Define $w = [v, u]$, $V = \frac{1}{2}w(uv + vu)w$, and consider the following elements of $\mathfrak{D}(L)$:

$$S = (V - \frac{1}{3}w^3)(V + \frac{1}{3}w^3)^{-1} + (V - \frac{1}{3}w^3)^{-1}(V + \frac{1}{3}w^3),$$

$$T = (w + v^2)^{-1}(w - v^2)S(w + v^2)(w - v^2)^{-1}.$$

The following hold true.

- (1) The k -subalgebra of $\mathfrak{D}(L)$ generated by S and T is the free k -algebra on the set $\{S, T\}$.
- (2) The elements S and T are symmetric with respect to the principal involution on $\mathfrak{D}(L)$.

Proof. Let L_1 be the Lie subalgebra of L generated by u and v .

Since $U(L)$ is an Ore domain, $U(L_1)$ is also an Ore domain. Moreover, $\mathfrak{D}(L_1) \subseteq \mathfrak{D}(L)$. Thus, we may suppose that L is generated by u and v .

For $l \leq 0$, let $U_l(L)$ be the k -subspace of $U(L)$ spanned by all the monomials in u and v of length $\leq -l$. In this way, we obtain a filtration

$$\cdots \supseteq U_l(L) \supseteq U_{l+1}(L) \supseteq \cdots \supseteq U_{-1}(L) \supseteq U_0(L) = k \supseteq 0 = U_1(L) \quad (4.1)$$

of $U(L)$. Let $\text{gr}(U(L))$ be the graded ring associated to this filtration:

$$\text{gr}(U(L)) = \bigoplus_{l \leq 0} U_l(L)/U_{l+1}(L).$$

Let $L_l = U_l(L) \cap L$, $l \leq 0$, be the induced filtration in L . We obtain in the natural way the graded Lie k -algebra $\text{gr}(L)$ associated to this filtration. It can be shown that $\text{gr}(L)$ generates $\text{gr}(U(L))$ as a k -algebra and that

$$U(\text{gr}(L)) \cong \text{gr}(U(L)) \quad (4.2)$$

under a natural isomorphism, i.e. the one extending $\text{gr}(L) \rightarrow \text{gr}(U(L))$ [16, Proposition 14] or [1, Lemma 2.1.2]. In what follows, the two objects in (4.2) will be identified.

As a first consequence of (4.2), we get that $\text{gr}(U(L))$ is a domain. Therefore the filtration (4.1) induces a valuation $\chi: U(L) \rightarrow \mathbb{Z} \cup \{\infty\}$ by $\chi(x) = l$ if $x \in U_l(L) \setminus U_{l+1}(L)$ and $\chi(0) = \infty$ [4, Proposition 2.6.1].

Now we can apply the results of Proposition 4.1 to $R = U(L)$. Consider the Laurent polynomial ring $U(L)[t, t^{-1}]$, and extend χ to a valuation, also denoted by χ , $U(L)[t, t^{-1}] \rightarrow \mathbb{Z} \cup \{\infty\}$, defined, as in Proposition 4.1(1), by

$$\chi\left(\sum_{i=l}^n t^i a_i\right) = \min_i \{\chi(a_i) + i\},$$

for all $\sum_{i=l}^n t^i a_i \in U(L)[t, t^{-1}]$.

Let $T = \{x \in U(L)[t, t^{-1}] \mid \chi(x) \geq 0\}$ and $T_0 = \{x \in U(L)[t, t^{-1}] \mid \chi(x) > 0\}$. Then there exists a ring isomorphism $\varphi: \text{gr}(U(L)) \rightarrow T/T_0$ by Proposition 4.1(2).

Consider u, v and $w = [v, u]$. Note that $u, v \in U_{-1}(L)$ and $w \in U_{-2}(L) \setminus U_{-1}(L)$ because L is not two-dimensional. In other words, $\chi(u) = \chi(v) = -1$ and $\chi(w) = -2$. Thus, also $u, v \in L_{-1}$ and $w \in L_{-2} \setminus L_{-1}$. Denote by \bar{u}, \bar{v} the class of $u, v \in U_{-1}(L)/U_0(L)$ and also the class of u and v in L_{-1}/L_0 . Denote by \bar{w} the class of w in $U_{-2}(L)/U_{-1}(L)$ and in L_{-2}/L_{-1} . Then $\varphi(\bar{u}) = tu + T_0$ and $\varphi(\bar{v}) = tv + T_0$ and $\varphi(\bar{w}) = t^2w + T_0$. By Proposition 4.1, T and $U(\text{gr}(L))$ are Ore domains. Let D be the Ore division ring of fractions of T and let Δ be the Ore division ring of fractions of $U(\text{gr}(L))$.

Now, $\text{gr}(L)$ is a (negatively) graded Lie k -algebra which is not commutative ($w \in L_{-2} \setminus L_{-1}$). Thus $\text{gr}(L)$ has to be a residually nilpotent Lie k -algebra. Observe that $[\bar{v}, \bar{u}] = \bar{w}$ as elements of $\text{gr}(L)$.

Now define $\bar{V} = \frac{1}{2}\bar{w}(\bar{u}\bar{v} + \bar{v}\bar{u})\bar{w}$,

$$\begin{aligned}\bar{S} &= (\bar{V} - \frac{1}{3}\bar{w}^3)(V + \frac{1}{3}\bar{w}^3)^{-1} + (\bar{V} - \frac{1}{3}\bar{w}^3)^{-1}(\bar{V} + \frac{1}{3}\bar{w}^3), \\ \bar{T} &= (\bar{w} + \bar{v}^2)^{-1}(\bar{w} - \bar{v}^2)\bar{S}(\bar{w} + \bar{v}^2)(\bar{w} - \bar{v}^2)^{-1}.\end{aligned}$$

Then Corollary 3.5(1) proves that \bar{S} and \bar{T} generate a free k -algebra in Δ .

Let $\mathcal{S} = T \setminus T_0$. By Proposition 4.1(4), \mathcal{S} is a left denominator set of T , and $\mathcal{S}^{-1}T$ is a local ring with maximal ideal $\mathcal{S}^{-1}T_0$. Moreover, there exists a morphism of rings $\psi: \mathcal{S}^{-1}T \rightarrow \Delta$ which induces the isomorphism $\mathcal{S}^{-1}T/\mathcal{S}^{-1}T_0 \cong \Delta$, by Proposition 4.1(5).

Consider $tu, tv, t^2w \in T \setminus T_0$. Observe that $\psi(tu) = \bar{u}$, $\psi(tv) = \bar{v}$ and $\psi(t^2w) = \bar{w}$.

Define $V' = \frac{1}{2}wt^2(utvt + vtut)wt^2 = t^6V$. Since u, v, w are k -linearly independent in L , the PBW-Theorem implies that u, v and $uv + vu = w + 2uv$ are k -linearly independent in $U(L)$. Hence $uv + vu \in U_{-2}(L) \setminus U_{-1}(L)$ and $\chi(uv + vu) = -2$. Therefore $\chi(V) = \chi(w(uv + vu)w) = -6$. By the definition of χ , $\chi(V') = 0$. Hence $V' \in T \setminus T_0$. In the same way, one can show that $V' - \frac{1}{3}(t^2w)^3 = t^6(V - \frac{1}{3}w^3) \in T \setminus T_0$, $V' + \frac{1}{3}(t^2w)^3 = t^6(V + \frac{1}{3}w^3) \in T \setminus T_0$, $t^2w + (tv)^2 = t^2(w + v^2) \in T \setminus T_0$ and $t^2w - (tv)^2 = t^2(w - v^2) \in T \setminus T_0$. Now define two elements S', T' of $\mathcal{S}^{-1}T$ as follows:

$$\begin{aligned}S' &= (V' - \frac{1}{3}(t^2w)^3)(V' + \frac{1}{3}(t^2w)^3)^{-1} + (V' - \frac{1}{3}(t^2w)^3)^{-1}(V' + \frac{1}{3}(t^2w)^3), \\ T' &= (t^2w + (tv)^2)^{-1}(t^2w - (tv)^2)S'(t^2w + (tv)^2)(t^2w - (tv)^2)^{-1}.\end{aligned}$$

Observe that $\psi(S') = \bar{S}$ and $\psi(T') = \bar{T}$. Therefore the k -algebra generated by $\{S', T'\}$ is the free k -algebra on $\{S', T'\}$.

And here comes magic. Since t is an element of the center, it follows that $T' = T$ and $S' = S$. Thus (1) is proved.

For (2), as in the proof of Theorem 2.6, it is not difficult to prove that S and T are symmetric with respect to the principal involution. \square

Remark 4.3. Let k be a field of characteristic zero. Let L be a Lie k -algebra generated by two noncommuting elements u and v such that $U(L)$ is an Ore domain. Suppose that L is of dimension at least three. By Lichtman's technique used in Theorem 4.2, the problem of finding a free k -algebra (generated by symmetric elements or not) inside $\mathfrak{D}(L)$ is reduced to the problem of finding a free Lie k -algebra inside $\mathfrak{D}(\text{gr}(L))$, where $\text{gr}(L)$ is a positively graded Lie k -algebra, $\text{gr}(L) = \bigoplus_{i=1}^{\infty} \text{gr}_i(L)$, which is not commutative and is generated by $\text{gr}_1(L)$. Then, by Corollary 3.5, $\mathfrak{D}(\text{gr}(L))$ contains a free k -algebra because it is a residually nilpotent Lie k -algebra

There is another way of obtaining a free k -algebra inside $\mathfrak{D}(L)$ as follows. We have the isomorphism $\frac{\text{gr}(L)}{\bigoplus_{i=3}^{\infty} \text{gr}_i(L)} \cong H$, the Heisenberg Lie k -algebra. By [18, Proposition 7.7] and [18, Corollary 7.2], there exists a specialization from $\mathfrak{D}(\text{gr}(L))$ to $\mathfrak{D}(H)$ extending the homomorphism of Lie k -algebras $\text{gr}(L) \rightarrow \frac{\text{gr}(L)}{\bigoplus_{i=3}^{\infty} \text{gr}_i(L)} \cong H$. Thus a suitable preimage of the free k -algebra obtained in Theorem 2.6 does the work. We would like to remark that our method of obtaining the free k -algebra inside $\mathfrak{D}(\text{gr}(L))$ is more elementary since [18,

Proposition 7.7] and [18, Corollary 7.2] rely on highly nontrivial facts. Moreover, our method is more general because it works for all classes of Lie k -algebras appearing in Examples 3.4, while this other method only works for positively graded Lie k -algebras (as $\text{gr}(L)$). \square

When the Lie subalgebra generated by u and v is of dimension two, we cannot apply the methods developed thus far, but we have the following consequence of Cauchon's Theorem.

Proposition 4.4. *Let k be a field of characteristic zero. Let M be the noncommutative two dimensional Lie k -algebra. Thus M has a basis $\{e, f\}$ such that $[e, f] = f$. Define $s = (e - \frac{1}{3})(e + \frac{1}{3})^{-1}$ and $u = (1 - f)(1 + f)^{-1}$. Consider the embedding $U(M) \hookrightarrow \mathfrak{D}(M)$. Then*

- (1) *the elements $s + s^{-1}$ and $u(s + s^{-1})u^{-1}$ are symmetric with respect to the principal involution;*
- (2) *the k -algebra generated by $\{s + s^{-1}, u(s + s^{-1})u^{-1}\}$ is the free k -algebra on $\{s + s^{-1}, u(s + s^{-1})u^{-1}\}$.*

Proof. Since $[e, f] = ef - fe = f$, $ef = f(e + 1)$. Thus $U(M)$ can be seen as a skew polynomial k -algebra, $U(M) = k[e][f; \sigma]$, where $\sigma(e) = e + 1$.

According to Cauchon's Theorem, if we define $s = (e - \frac{1}{3})(e + \frac{1}{3})^{-1}$ and $u = (1 - f)(1 + f)^{-1}$, the elements s and usu^{-1} generate a free group k -algebra. Then Corollary 2.2 implies that the k -algebra generated by $\{s + s^{-1}, u(s + s^{-1})u^{-1}\}$ is the free k -algebra on $\{s + s^{-1}, u(s + s^{-1})u^{-1}\}$. Moreover

$$s^* = \left((e - \frac{1}{3})(e + \frac{1}{3})^{-1} \right)^* = (-e + \frac{1}{3})^{-1}(-e - \frac{1}{3}) = (e + \frac{1}{3})(e - \frac{1}{3})^{-1} = s^{-1},$$

$$u^* = ((1 - f)(1 + f)^{-1})^* = (1 - f)^{-1}(1 + f) = (1 + f)(1 - f)^{-1} = u^{-1}.$$

Therefore $s + s^{-1}$ and $u(s + s^{-1})u^{-1}$, are symmetric. \square

5. FURTHER COMMENTS

Let k be a field of characteristic zero, L be a noncommutative Lie k -algebra and $U(L)$ be its universal enveloping algebra.

Until now, we have proved that if $u, v \in L$ with $[u, v] \neq 0$ and K is the Lie k -subalgebra they generate, then $\mathfrak{D}(K) (\subseteq \mathfrak{D}(L))$ contains a free k -algebra generated by symmetric elements with respect to the principal involution of $\mathfrak{D}(L)$ and we give explicit generators of this free k -algebra, provided that either $U(K)$ is an Ore domain or there exists a morphism of Lie k -algebras $K \rightarrow H$, $u \mapsto x$, $v \mapsto y$, where H is the Heisenberg Lie k -algebra.

Moreover, if L is a Lie k -algebra that contains two elements u, v such that the Lie k -algebra generated by them is the free Lie k -algebra on the set $\{u, v\}$, then $\mathfrak{D}(L)$ contains a free k -algebra generated by symmetric elements with respect to the principal involution. Indeed, let F be the free Lie k -algebra generated by $\{u, v\}$. Recall that $U(F)$, the universal enveloping algebra of F , is the free k -algebra $k\langle u, v \rangle$. Hence u^2 and v^2 are symmetric, and the k -algebra they generate is the free k -algebra $k\langle u^2, v^2 \rangle \subseteq U(F) \subseteq U(L) \subseteq \mathfrak{D}(L)$. (Note that, in this case, the characteristic of k need not be zero.)

Now we raise the following natural question in the case of a field k of characteristic zero. What do possible counterexamples of Lie k -algebras L such that $\mathfrak{D}(L)$ does not contain a free noncommutative k -algebra generated by symmetric elements look like?

We assert that such a possible counterexample cannot be of subexponential growth. Indeed, by [25], if L is a Lie k -algebra of subexponential growth, then $U(L)$ is of subexponential growth. Then, since a noncommutative free k -algebra is of exponential growth, we obtain that $U(L)$ does not contain a noncommutative free k -algebra and, therefore, $U(L)$ is an Ore domain because any k -algebra which is not an Ore domain must contain a noncommutative free k -algebra [13, Proposition 10.25]. Now recall that if L_1 is a Lie subalgebra of L , then $U(L_1)$ is an Ore domain (because $U(L)$ is). Thus the claim is proved.

Therefore L must be of exponential growth but cannot contain a noncommutative free Lie k -algebra. Also, L must satisfy the following properties: for any two noncommuting elements $u, v \in L$, if we denote by K the Lie k -subalgebra they generate, then K cannot be of subexponential growth, and K cannot be mapped onto H (in particular K can neither be residually nilpotent nor have a central series). Although we think that such examples could exist, we have not been able to find any.

The next theorem is the main result of [16]. We show that the results obtained thus far also prove it.

Theorem 5.1. *Let k be a field of characteristic zero and L be a noncommutative Lie k -algebra. Then any division ring that contains the universal enveloping algebra $U(L)$ of L contains a noncommutative free k -algebra.*

Proof. It is well known that if $U(L)$ is not Ore, then it contains a noncommutative free k -algebra [13, Proposition 10.25].

If $U(L)$ is an Ore domain, then $\mathfrak{D}(L)$ is the Ore division ring of fractions of $U(L)$ and it is contained in any division ring that contains $U(L)$. Then Theorem 4.2 and Proposition 4.4 imply that $\mathfrak{D}(L)$ contains a free k -algebra. \square

Let now k be an uncountable field. Suppose that D is a division ring that contains k as a central subfield. Suppose that there exist different elements S and T in D that generate a free k -algebra $k\langle S, T \rangle$. It was shown in [8] that there exist $a, b \in k$ such that the k -algebra generated by $\{1 + aS, (1 + aS)^{-1}, 1 + bT, (1 + bT)^{-1}\}$ is the free group k -algebra on the set $\{1 + aS, 1 + bT\}$.

Let now L be a Lie k -algebra with k uncountable and of characteristic zero. Suppose that L is one of the Lie k -algebras for which we have proved that $\mathfrak{D}(L)$ contains two symmetric elements with respect to the principal involution S and T such that the k -algebra they generate is free on $\{S, T\}$. By the foregoing paragraph, there exist $a, b \in k$ such that the k -algebra generated by $\{1 + aS, (1 + aS)^{-1}, 1 + bT, (1 + bT)^{-1}\}$ is the free group k -algebra on the set $\{1 + aS, 1 + bT\}$. Observe that $1 + aS$ and $1 + bT$ are also symmetric with respect to the principal involution.

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